

On Rayleigh's Stability Criterion for Couette Flow

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The stability problem of a stationary circular flow of an ideal fluid between two coaxial cylinders is considered. It is shown that Rayleigh's circulation criterion is necessary and sufficient for the total kinetic energy of an axisymmetric disturbance to be bounded in time.

The flow in an ideal fluid is described by Euler's system of equations

$$\operatorname{div} \mathbf{v} = 0, \quad (1)$$

$$\dot{\mathbf{v}} + \frac{1}{2} \nabla v^2 - \mathbf{v} \times \operatorname{curl} \mathbf{v} = -\frac{1}{\rho} \nabla p, \quad (2)$$

where the dot is the partial time derivative and the other symbols have their usual meaning. If the vector field \mathbf{v} and the scalar p are axisymmetric, then the general solution of (1) can be written in the form

$$\mathbf{v} = \nabla \varphi \times \nabla \psi + v_\varphi r \nabla \varphi, \quad (3)$$

where r, φ, z are cylindrical coordinates. In representation (3) the stream function ψ and the toroidal velocity v_φ are axisymmetric scalars, i.e. $\psi, v_\varphi(r, z, t)$. Representation (3) yields the vorticity

$$\mathbf{w} = \operatorname{curl} \mathbf{v} = (\Delta_* \psi) \nabla \varphi - (\nabla \varphi) \times \nabla r v_\varphi, \quad (4)$$

where

$$\Delta_* = r \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

is the Stokes operator. If the pressure $p(r)$ is determined correspondingly, then

$$\psi \equiv 0, \quad v_\varphi = V(r) \quad (5)$$

is a stationary solution of (1) and (2), where the profile function $V(r)$ can be chosen arbitrarily. Let us suppose that $V(r)$ and its derivative $V'(r)$ are bounded:

$$|V| \leq C_1, \quad |V'| \leq C_2. \quad (6)$$

Let $\tilde{v}_\varphi = v(r, z, t)$ be the deviation of V . Then the linearization in Ψ and v yields for the φ -component of

(2) and the φ -component of the curl of (2), respectively,

$$\dot{v} + \frac{\Phi}{2V} \frac{\partial \psi}{\partial z} = 0, \quad (7)$$

$$\Delta_* \psi - 2V \frac{\partial v}{\partial z} = 0, \quad (8)$$

where

$$\Phi = \frac{1}{r^3} (r^2 V^2)' = 2 \frac{V}{r} \left(V' + \frac{V}{r} \right)$$

is the Rayleigh discriminant (see, for instance, [1]). Equations (7) and (8) are investigated for the intervals

$$R_1 \leq r \leq R_2, \quad 0 \leq z \leq L$$

together with the boundary conditions

$$\psi(R_1, z, t) = \psi(R_2, z, t) = 0, \quad (9)$$

$$\psi, v \text{ periodic in } z \text{ with period } L. \quad (10)$$

A profile $V(r)$ is called stable if for all admissible initial conditions

$$\psi(r, z, 0) = \psi^*(r, z), \quad v(r, z, 0) = v^*(r, z)$$

(7)–(10) have only solutions for which the total kinetic energy of the disturbances is bounded in time. On the other hand, if there are admissible initial conditions such that (7)–(10) have solutions whose total kinetic energy is unbounded in time, then we call the profile $V(r)$ unstable.

Let us first consider the case where Φ is positive. We then have

$$0 < C_3 \leq \Phi \leq C_4. \quad (11)$$

In this case the circulation of the stationary flow increases outwardly. It then follows from conditions (6) that V cannot vanish:

$$|V| \geq C_5.$$

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Let us now multiply (7) by $4V^2v/(r^2\Phi)$, (8) by $-\psi/r^2$, and integrate over space. Adding the results leads to

$$\left\langle \tilde{v}_r^2 + \tilde{v}_z^2 + \frac{4V^2}{r^2\Phi} v_\phi^2 \right\rangle = 0, \quad (12)$$

where

$$\tilde{v}_r = \frac{1}{r} \frac{\partial \psi}{\partial z}, \quad \tilde{v}_z = -\frac{1}{r} \frac{\partial \psi}{\partial r}$$

and

$$\langle \dots \rangle = 2\pi \int_{R_1}^{R_2} \int_0^L \dots r \, dr \, dz$$

is the volume integral. Equation (12) can be integrated in time to give an integral of motion which has a lower bound

$$C_6 = \left\langle \tilde{v}_r^2 + \tilde{v}_z^2 + \frac{4V^2}{r^2\Phi} \tilde{v}_\phi^2 \right\rangle \geq \langle \tilde{v}_r^2 + \tilde{v}_z^2 + C_7 \tilde{v}_\phi^2 \rangle,$$

where C_7 is the minimum of $4V^2/(r^2\Phi)$. This shows that in the case of positive Φ the total kinetic energy of the disturbance is bounded in time.

In order to show instability, v is eliminated from (7) and (8) to give

$$\Delta_* \ddot{\psi} + \Phi \frac{\partial^2 \psi}{\partial z^2} = 0. \quad (13)$$

Analogously to the stability problem of the atmosphere [2] and to ideal magnetohydrodynamics [3], the three functionals

$$K(\psi) = -\frac{1}{2} \left\langle \frac{1}{r^2} \dot{\psi} \Delta_* \dot{\psi} \right\rangle,$$

$$I(\psi) = -\frac{1}{2} \left\langle \frac{1}{r^2} \psi \Delta_* \psi \right\rangle, \quad W(\psi) = \frac{1}{2} \left\langle \frac{\Phi}{r^2} \left(\frac{\partial \psi}{\partial z} \right)^2 \right\rangle$$

are defined. For the boundary conditions (9), (10) both differential operators in (13) are symmetric, and K and I are positive definite. W is positive semi-definite if the function Φ is positive. Suppose now that there is an interval $r_1 \leq r \leq r_2$, where Φ is negative:

$$\Phi < 0 \quad \text{for} \quad r_1 \leq r \leq r_2. \quad (14)$$

Then

$$w(r, z) = U(r) \sin \frac{2\pi z}{L}$$

is an admissible test function with $W(w) < 0$ if $U(r) \neq 0$ in the interval $r_1 \leq r \leq r_2$ and $U(r) \equiv 0$ outside the interval ("localization"). Multiplication of (13) by $\dot{\psi}/r^2$ and ψ/r^2 and spatial integration yields, respectively, the two functional equations

$$K(\psi) + W(\psi) = C_8, \quad \dot{I}(\psi) = 2[K(\psi) - W(\psi)].$$

It is now possible to choose the initial conditions

$$\psi(r, z, 0) = w(r, z), \quad \dot{\psi}(r, z, 0) = \gamma w(r, z)$$

with

$$\gamma^2 = -\frac{W(w)}{I(w)} > 0.$$

We then get $C_8 = 0$, and it can be shown [3] that the estimate

$$I(\psi(t)) \geq I(w) e^{2\gamma t}$$

is valid. This means that the poloidal part of the total kinetic energy grows exponentially in time because

$$I(\psi) = \frac{1}{2} \langle \tilde{v}_r^2 + \tilde{v}_z^2 \rangle.$$

Finally, it remains to discuss the case that there is an interval where Φ vanishes identically, while it is non-negative outside this interval. In this case there are solutions of the form

$$\psi = \hat{\psi}(r, z) t, \quad v = \hat{v}(r, z),$$

which are localized to the interval where $\Phi \equiv 0$ and are connected by

$$\Delta_* \hat{\psi} - 2V \frac{\partial \hat{v}}{\partial z} = 0.$$

This is possible if the z -average of $\hat{\psi}$ vanishes. So, in this case the total kinetic energy also increases unboundedly in time. This increase, however, is only algebraic.

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- [1] P. G. Drazin and W. H. Reid, *Hydrodynamic Stability*, Cambridge University Press, Cambridge 1981.
 [2] D. Lortz, *Z. Naturforsch.* **26a**, 1992 (1971). On Schwarzschild's stability criterion.

- [3] G. Laval, C. Mercier, and R. Pellat, Necessity of the Energy Principle for Magnetohydrostatic Stability, *Nuclear Fusion* **5**, 156–158 (1965).